

AN EXPLICIT APPROACH TO THE AHLGREN-ONO CONJECTURE

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ABSTRACT. Let $p(n)$ be the partition function. Ahlgren and Ono conjectured that every arithmetic progression contains infinitely many integers N for which $p(N)$ is not congruent to 0 (mod 3). Radu proved this conjecture in 2010 using work of Deligne and Rapoport. In this note, we give a simpler proof of Ahlgren and Ono's conjecture in the special case where the modulus of the arithmetic progression is a power of 3 by applying a method of Boylan and Ono and using work of Bellaïche and Khare generalizing Serre's results on the local nilpotency of the Hecke algebra.

1. INTRODUCTION AND STATEMENT OF RESULTS

A *partition* of a nonnegative integer n is a non-increasing sequence of positive integers whose sum is n . The partition function $p(n)$ then counts the number of distinct partitions of n . The generating function for $p(n)$ was shown by Euler to be

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots.$$

Ramanujan famously observed the congruences $p(5k+4) \equiv 0 \pmod{5}$, $p(7k+5) \equiv 0 \pmod{7}$, and $p(11k+6) \equiv 0 \pmod{11}$. More recently, Ono in [9] and Folsom, Kent, and Ono in [5] have used Serre's theory of modular forms modulo p to prove general results about the p -adic behavior of $p(n)$ for all $p \geq 5$. As a consequence, the behavior of $p(n)$ is well understood modulo p for all $p \geq 5$.

The behavior of $p(n)$ modulo 2 and 3 is far less well understood. It is widely believed that $p(n)$ is equidistributed modulo 2 and 3, but little is known. Subbarao conjectured in [12] that for any arithmetic progression $B \pmod{A}$, there are infinitely many integers $N \equiv B \pmod{A}$ for which $p(N)$ is even, and also infinitely many such N for which $p(N)$ is odd. In [1], Ahlgren and Ono conjectured that for any arithmetic progression $B \pmod{A}$, there are infinitely many integers $N \equiv B \pmod{A}$ for which $p(N) \not\equiv 0 \pmod{3}$.

Ono [8] established half of Subbarao's conjecture, proving that there are infinitely many N in every arithmetic progression for which $p(n)$ is even. Boylan and Ono [3] then used the local nilpotency of the Hecke algebra, as observed by Serre in [11, p. 115], to prove the odd case of Subbarao's conjecture in the case $A = 2^s$. Radu proves the full conjecture, along with Ahlgren and Ono's for modulo 3, in [10], using work [4] of Deligne and Rapoport that applies the structure of the Tate curve to study the Fourier coefficients of modular forms.

In this note, we adapt the method of Boylan and Ono to provide a simpler, more explicit proof of Ahlgren and Ono's conjecture in the case $A = 3^s$. We rely on Bellaïche and Khare's generalization [2] of Serre's explicit description [6, 7] of the Hecke algebra for modulo 2 reductions of level 1 modular forms. We are then able to show the following.

Theorem 1.1. *Let s and a be positive integers. There are infinitely many positive integers $n \equiv a \pmod{3^s}$ such that $p(n) \not\equiv 0 \pmod{3}$.*

In Section 2, we discuss Bellaïche and Khare's work. In Section 3, we prove Theorem 1.1.

2. LOCAL NILPOTENCY

Throughout, let $\Delta = \Delta(z)$ denote the discriminant modular form, and $\tau(n)$ denote the Fourier coefficients of Δ . Let p be a prime number, and let T_p be the p th Hecke operator, which by definition acts on a modular form $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ of weight k by

$$T_p f(z) = \sum_{n=0}^{\infty} (a(np) + p^{k-1}a(n/p))q^n$$

where $a(x) = 0$ if $x \notin \mathbb{Z}$. In [6], Nicolas and Serre compute for each f the minimal integer $g = g(f)$ such that for any g primes p_1, \dots, p_g , we have

$$T_{p_1} \cdots T_{p_g} f(z) \equiv 0 \pmod{2}.$$

The statement that the algebra generated by the Hecke operators is locally nilpotent modulo 2 just means that $g(f)$ is finite for every f . Additionally, Nicolas and Serre prove in [7] that each T_p can be written as a power series in T_3 and T_5 , thereby permitting the enumeration of all sets of primes p_1, \dots, p_{g-1} such that $T_{p_1} \cdots T_{p_{g-1}} f(z)$ is *not* zero modulo 2. As a result of this description of the Hecke algebra modulo 2, Boylan and Ono's method in [3] becomes completely explicit.

Since our goal is to recreate Boylan and Ono's work in the modulo 3 case, we replace Nicolas and Serre's conclusions with the following extension by Bellaïche and Khare. Write $T'_p = T_p$ if $p \equiv 2 \pmod{3}$, and $T'_p = 1 + T_p$ if $p \equiv 1 \pmod{3}$. Then the operators T'_p act locally nilpotently on the ring $S(\mathbb{F}_3)$ of level 1 cusp forms with integer coefficients taken modulo 3, in the sense that given such a modular form f , there is some minimal integer g such that $T'_{p_1} \cdots T'_{p_g} f = 0$ for any sequence of Hecke operators $T'_{p_1}, \dots, T'_{p_g}$. As such, since Δ and $-\Delta$ are the only cusp forms modulo 3 satisfying $T'_p | f = 0$ for all p , we have some maximal sequence p_1, \dots, p_{g-1} such that $T'_{p_1} \cdots T'_{p_{g-1}} f = \pm \Delta$. Moreover, we have the following description of the Hecke algebra on modular forms with coefficients in \mathbb{F}_3 .

Theorem 2.1 ([2], Theorem 24). *The algebra of Hecke operators on $S(\mathbb{F}_3)$ is isomorphic to the power series ring $\mathbb{F}_3[[x, y]]$, with an isomorphism given by sending $T'_2 = T_2$ to x and $T'_7 = 1 + T_7$ to y . Assuming this identification, we have $T'_p \equiv x \pmod{(x, y)^2}$ if and only if p is congruent to 2 (mod 3) but not 8 (mod 9), $T'_p \equiv y \pmod{(x, y)^2}$ if and only if p is congruent to 1 (mod 3) and not split in the splitting field of $X^3 - 3$, and otherwise $T'_p \equiv 0 \pmod{(x, y)^2}$.*

This theorem in particular implies that for any nonzero $f \in S(\mathbb{F}_3)$ there are some positive integers k and ℓ such that $(T'_2)^k (T'_7)^\ell f = \pm \Delta$. We may now proceed according to Boylan and Ono's strategy.

3. PARTITIONS MODULO 3

In this section we prove Theorem 1.1. We start by proving a basic lemma, similar to Corollary 1.4 of [10].

Lemma 3.1. *Suppose that for every s , every arithmetic progression modulo 3^s contains at least one N such that $p(N) \not\equiv 0 \pmod{3}$. Then for every s , every arithmetic progression modulo 3^s contains infinitely many such N .*

Proof. We prove the contrapositive. Suppose there exist some r and s , with $0 \leq r < 3^s$, for which there are only finitely many $N \equiv r \pmod{3^s}$ with $p(N) \not\equiv 0 \pmod{3}$. Then there is some $a_0 \geq 0$ such that $p(3^s a + r) \equiv 0 \pmod{3}$ for all $a \geq a_0$. Let t be such that $3^t > 3^s a_0 + r$. Then we have

$$p(3^s \cdot 3^t a + 3^s a_0 + r) \equiv 0 \pmod{3}$$

for all $a \geq 0$, from which we conclude that $p(N) \equiv 0 \pmod{3}$ for all $N \equiv 3^s a_0 + r \pmod{3^{s+t}}$. \square

Let the integers $a_s(n)$ be defined by the generating function $\sum_{n=0}^{\infty} a_s(n) q^n = \Delta^{\frac{9^s-1}{8}}$, and let $r_s(j)$ be defined by $1 + \sum_{n=1}^{\infty} r_s(j) q^{3 \cdot 9^s j} = \prod_{n=1}^{\infty} (1 - q^{3 \cdot 9^s n})$. Our next lemma is similar to Lemma 2.1 of [3].

Lemma 3.2. *We have*

$$a_s(n) \equiv p\left(\frac{n - \frac{9^s-1}{8}}{3}\right) + \sum_{j=1}^{\infty} r_s(j) p\left(\frac{n - \frac{9^s-1}{8}}{3} - 9^s j\right) \pmod{3}.$$

Proof. We may compute

$$\begin{aligned} \Delta^{\frac{9^s-1}{8}} &= \left(q \prod_{n=1}^{\infty} (1 - q^n)^{24} \right)^{\frac{9^s-1}{8}} \\ &\equiv q^{\frac{9^s-1}{8}} \prod_{n=1}^{\infty} (1 - q^{3 \cdot 9^s n}) \prod_{n=1}^{\infty} \frac{1}{1 - q^{3n}} \pmod{3} \\ &\equiv \left(1 + \sum_{j=1}^{\infty} r_s(j) q^{3 \cdot 9^s j} \right) \left(\sum_{k=0}^{\infty} p(k) q^{3k + \frac{9^s-1}{8}} \right) \pmod{3}. \end{aligned}$$

The lemma follows by comparing coefficients. \square

Finally, we require the following lemma concerning nonzero coefficients of $\Delta^{\frac{9^s-1}{8}}$.

Lemma 3.3. *There are fixed nonnegative integers k and ℓ , not both zero, such that the following holds. Let p_1, \dots, p_k be distinct primes which satisfy $p_i \equiv 2 \pmod{3}$ and $p_i \not\equiv 8 \pmod{9}$ for all i . Let q_1, \dots, q_ℓ be distinct primes such that for all j we have $q_j \equiv 1 \pmod{9^{s+1}}$ and q_j does not split in the splitting field of $X^3 - 3$. Then for any n_0 satisfying $(n_0, p_1 \cdots p_k q_1 \cdots q_\ell) = 1$ such that $\tau(n_0) \not\equiv 0 \pmod{3}$, there is some $d | q_1 \cdots q_\ell$ for which we have $a_s(n_0 p_1 \cdots p_k d) \not\equiv 0 \pmod{3}$.*

Remark. Given values of k , ℓ , and s , it is always possible to find corresponding primes p_1, \dots, p_k and q_1, \dots, q_ℓ ; indeed by the Chebotarev density theorem, the primes that are valid choices for p_i have density $1/3$ within the primes, and the primes that are valid choices for q_j have density $1/(2 \cdot 3^{s+1})$.

Proof of Lemma 3.3. Identify the Hecke algebra modulo 3 with the ring $\mathbb{F}_3[[x, y]]$ as described in Theorem 2.1 and let $p_1, \dots, p_k, q_1, \dots, q_\ell$ be such that $T'_{p_i} \equiv x \pmod{(x, y)^2}$ and $T'_{q_j} \equiv y \pmod{(x, y)^2}$, and that

$$T'_{p_1} \cdots T'_{p_k} T'_{q_1} \cdots T'_{q_\ell} \Delta(z)^{\frac{9^s-1}{8}} \equiv \pm \Delta \pmod{3}.$$

Note that having fixed k and ℓ so that such p_i s and q_j s exist, any such sequence of primes satisfying the hypotheses of Lemma 3.3 satisfy the same equation. By comparing coefficients, we conclude that for any n_0 not divisible by the p_i s and q_j s with a nonzero corresponding coefficient of Δ , we have

$$(1) \quad \sum_{d|q_1 \cdots q_\ell} a_s(n_0 p_1 \cdots p_k q_1 \cdots q_\ell) \not\equiv 0 \pmod{3}.$$

Lemma 3.3 follows. \square

Finally, combining Lemma 3.3 with Lemmas 3.1 and 3.2, we prove Theorem 1.1.

Deduction of Theorem 1.1 from Lemma 3.3. Choose k and ℓ such that $(T'_2)^k (T'_7)^\ell \Delta^{\frac{9^s-1}{8}} = \pm \Delta$ let $p_1, \dots, p_k, q_1, \dots, q_\ell$ be as in the statement of Lemma 3.3. Then, by Lemma 3.3, given any n_0 with $\tau(n_0) \not\equiv 0 \pmod{3}$ and $(n_0, p_1 \cdots p_k q_1 \cdots q_\ell) = 1$ we have some $d|q_1 \cdots q_\ell$ such that $a_s(n_0 p_1 \cdots p_k d) \not\equiv 0 \pmod{3}$. By Lemma 3.2, for this $d|q_1 \cdots q_\ell$, we may write

$$a_s(n_0 p_1 \cdots p_k d) \equiv p \left(\frac{n_0 p_1 \cdots p_k d - \frac{9^s-1}{8}}{3} \right) + \sum_{j=1}^{\infty} r_s(j) p \left(\frac{n_0 p_1 \cdots p_k d - \frac{9^s-1}{8}}{3} - 9^s j \right) \pmod{3}.$$

Since $a_s(p_1 \cdots p_k d) \not\equiv 0 \pmod{3}$, we conclude that $p \left(\frac{p_1 \cdots p_k d - \frac{9^s-1}{8}}{3} - 9^s j \right) \not\equiv 0 \pmod{3}$ for

some $j \geq 0$, so for some n satisfying $n \equiv \frac{p_1 \cdots p_k d - \frac{9^s-1}{8}}{3} \pmod{9^s}$ we have $p(n) \not\equiv 0 \pmod{3}$.

Hence, by Lemma 3.1 it suffices to show that as we vary n_0 the quantity $\frac{n_0 p_1 \cdots p_k d - \frac{9^s-1}{8}}{3}$ covers all residue classes modulo 9^s . Since $d \equiv 1 \pmod{9^{s+1}}$ regardless of its precise factorization, it may be dropped. To show that $\frac{n_0 p_1 \cdots p_k - \frac{9^s-1}{8}}{3}$ covers all residue classes as we vary n_0 , we note first note the standard congruence for the τ function,

$$\tau(n) \equiv \sigma_1(n) \pmod{3},$$

is valid for all $n \not\equiv 0 \pmod{3}$, and in particular $\tau(p) \equiv 2 \pmod{3}$ when $p \equiv 1 \pmod{3}$. In addition, we note that k must be even, since $a_s(n) \not\equiv 0 \pmod{3}$ implies $n \equiv 1 \pmod{3}$, but if k is odd we have $a_s(p_1 \cdots p_k D) \not\equiv 0 \pmod{3}$ even though $p_1 \cdots p_k D \equiv 2 \pmod{3}$, a contradiction. So we have that $p_1 \cdots p_k \equiv 1 \pmod{3}$, so to show that the terms $\frac{n_0 p_1 \cdots p_k - \frac{9^s-1}{8}}{3}$ cover all residue classes we need only find n_0 in all residue classes $n \pmod{9^{s+1}}$ satisfying $n \equiv 1 \pmod{3}$. But to do so we may simply take n_0 to be any prime satisfying $n_0 \equiv n \pmod{9^{s+1}}$, of which there are infinitely many. Theorem 1.1 follows by applying Lemma 3.1. \square

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